Sydney Girls High School
2023
Trial Higher School Certificate
Examination

Mathematics Extension 1

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using a black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- In Questions 11-14, show relevant mathematical reasoning and/or calculations

Total marks: 70

Section I – 10 marks (pages 3-7)

- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 8-14)

- Attempt Questions 11-14
- Allow about 1 hour and 45 minutes for this section

Name: …………………………………………………… Teacher: ……………………………………………………	**THIS IS A TRIAL PAPER ONLY** It does not necessarily reflect the format or the content of the 2023 HSC Examination Paper in this subject.

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Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the multiple choice answer sheet for Questions 1 – 10.

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1 What is the remainder when $P(x) = 2x^3 - 3x + 2$ is divided by $x + 2$?

A. 12

B. -8

C. 2

D. -3

2 What is the angle between the vectors $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -7 \end{pmatrix}$?

A. 30°

B. 45°

C. 135°

D. 150°

3 Which of the following is equal to $\int \sin^2 2x \; dx$?

A. $\frac{1}{2}x + \frac{1}{8}\sin 4x + C$

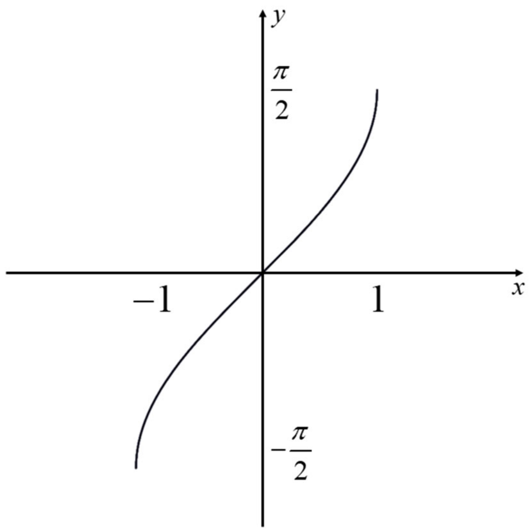
B. $\frac{1}{2}x - \frac{1}{8}\sin 4x + C$

C. $\frac{1}{2}x + \frac{1}{8}\cos 4x + C$

D. $\frac{1}{2}x - \frac{1}{8}\cos 4x + C$

4 Which of the following curves represents $y = 3\sin^{-1} 2x$?

A.



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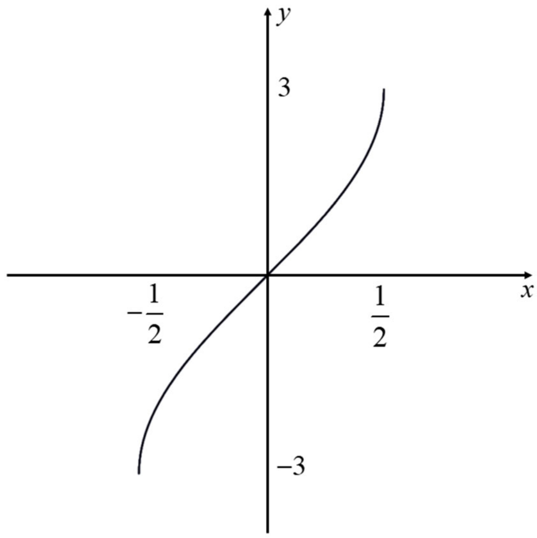

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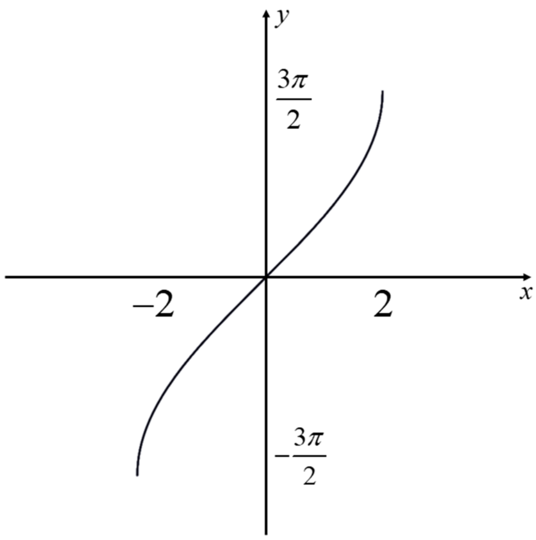
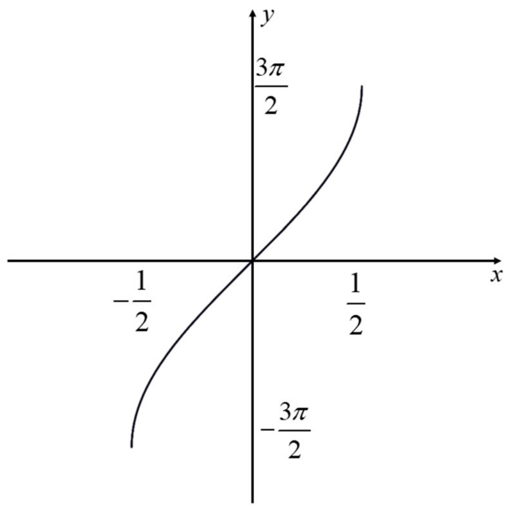


D.



5 Which of the following is equal to $\displaystyle\int_0^{\sqrt{3}} \frac{1}{\left(x^2+1\right)^2}\,dx$ after applying the substitution $x = \tan\theta$?

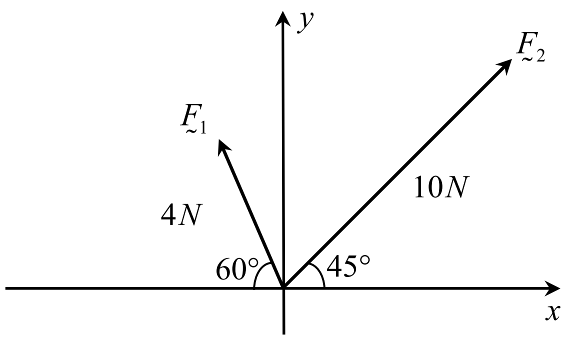
A. $\displaystyle\int_0^{\frac{\pi}{3}} \cos^2\theta\ d\theta$

B. $\displaystyle\int_0^{\frac{\pi}{3}} \sec^2\theta\ d\theta$

C. $\displaystyle\int_0^{\frac{\pi}{6}} \cos^2\theta\ d\theta$

D. $\displaystyle\int_0^{\frac{\pi}{6}} \sec^2\theta\ d\theta$

6 Two forces act simultaneously on a particle situated at the origin, as shown.



What is the resultant force, $\underset{\sim}{F}_1 + \underset{\sim}{F}_2$?

A. $\left(2+5\sqrt{2}\right)\underset{\sim}{i}+\left(-2\sqrt{3}+5\sqrt{2}\right)\underset{\sim}{j}$

B. $\left(-2+5\sqrt{2}\right)\underset{\sim}{i}+\left(-2\sqrt{3}+5\sqrt{2}\right)\underset{\sim}{j}$

C. $\left(-2+5\sqrt{2}\right)\underset{\sim}{i}+\left(2\sqrt{3}+5\sqrt{2}\right)\underset{\sim}{j}$

D. $\left(2+5\sqrt{2}\right)\underset{\sim}{i}+\left(2\sqrt{3}+5\sqrt{2}\right)\underset{\sim}{j}$

7 The displacement of a body is given by $x = 2 - e^{-4t}$.

Which of the following is the acceleration of the body?

A. $a = 16(x+2)$

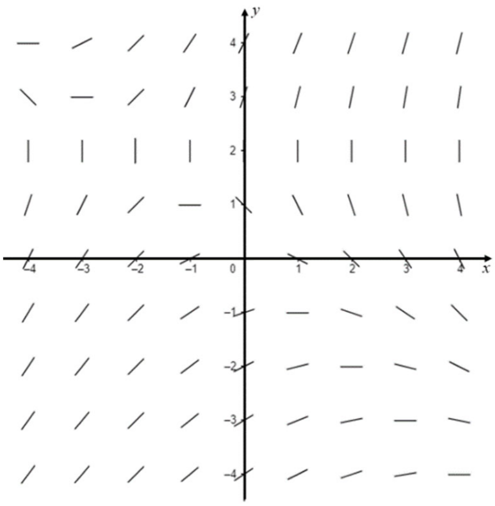
B. $a = 16(x-2)$

C. $a = -16(x+2)$

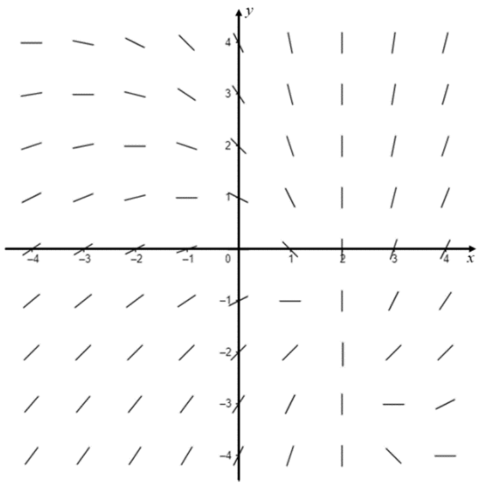
D. $a = -16(x-2)$

8 Which of the following represents the slope field of $\dfrac{dy}{dx} = \dfrac{x+y}{y-2}$?

A.



B.



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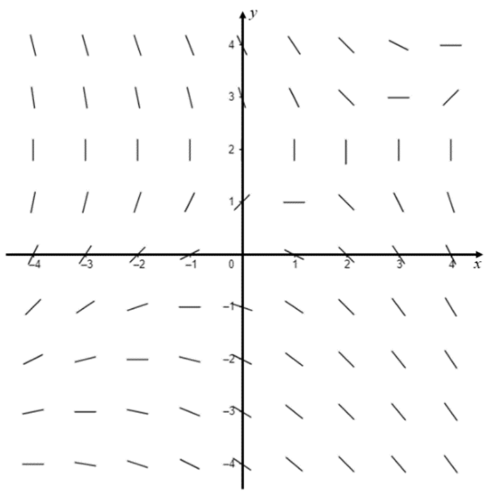
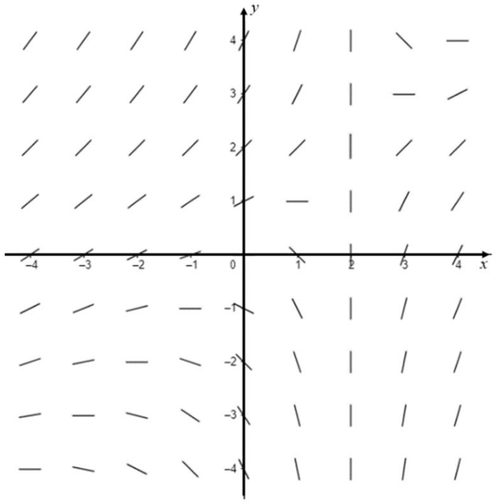


D.



9 Four wombats and three possums nap in a circle.

In how many ways can these animals be arranged around the circle if no two possums nap next to each other?

A. 36

B. 144

C. 540

D. 720

10 The vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$ have identical magnitudes.

Which of the following is a correct expression for the projection of $\underset{\sim}{u}$ onto $\underset{\sim}{v}$?

A. $\left(\underset{\sim}{u} \cdot \underset{\sim}{v}\right) \underset{\sim}{v}$

B. $\left(\underset{\sim}{\hat{u}} \cdot \underset{\sim}{v}\right) \underset{\sim}{v}$

C. $\left(\underset{\sim}{\hat{u}} \cdot \underset{\sim}{\hat{v}}\right) \underset{\sim}{v}$

D. $\left(\underset{\sim}{\hat{u}} \cdot \underset{\sim}{\hat{v}}\right) \underset{\sim}{\hat{v}}$

Section II

60 marks
Attempt Questions 11 – 14
Allow about 1 hour and 45 minutes for this section

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Start a new page.

(a) Evaluate $\int_0^{\frac{3}{\sqrt{2}}} \frac{dx}{\sqrt{9-x^2}}$. **2**

(b) Express $3\cos x - \sqrt{3}\sin x$ in the form $R\cos(x+\alpha)$. **3**

(c) Find the value of the constant term in the expansion of $\left(x - \frac{2}{x^2}\right)^6$. **2**

(d) Tea is spilling onto the floor, forming an expanding circle. **2**

The radius of this circle increases at a rate of 2 cm/s.

At what rate is the surface area of the tea increasing when the radius is 8 cm?

Question 11 continues on the following page

Question 1 (continued)

(e) A team of 12 basketball players is selected from a pool of 8 left-handed students and 16 right-handed students. How many ways can the team be chosen if:

(i) There are no restrictions? **1**

(ii) The team has more left-handed students than right-handed students? **2**

(f) Solve $\dfrac{3}{3-2x} \geq x+1$. **3**

End of Question 11

Question 12 (15 marks) Start a new page.

(a) A population of bacteria declines rapidly according to the equation

$$\frac{dB}{dt} = -kB,$$

where B is the number of bacteria after t seconds, and k is a positive constant.

(i) Show that $B = B_0 e^{-kt}$ is a solution to the equation, where B_0 is a constant. **1**

(ii) Find the value of k given that it takes 7 seconds for 99% of the initial bacteria population to die. Leave your answer in exact form. **3**

(b) Find the Cartesian equation of the curve given by: **2**

$$x = 3\cos\theta \quad \text{and} \quad y = 3\sin\theta - 5.$$

(c) (i) Write $\sin\theta + \sin 3\theta$ as a product of trigonometric functions. **1**

(ii) Hence, show that $\dfrac{\sin\theta + \sin 3\theta}{\cos 2\theta + 1} = 2\sin\theta$. **2**

(d) Find $\displaystyle\int \frac{x-2}{\sqrt{x+2}}\, dx$ using the substitution $u = x + 2$. **3**

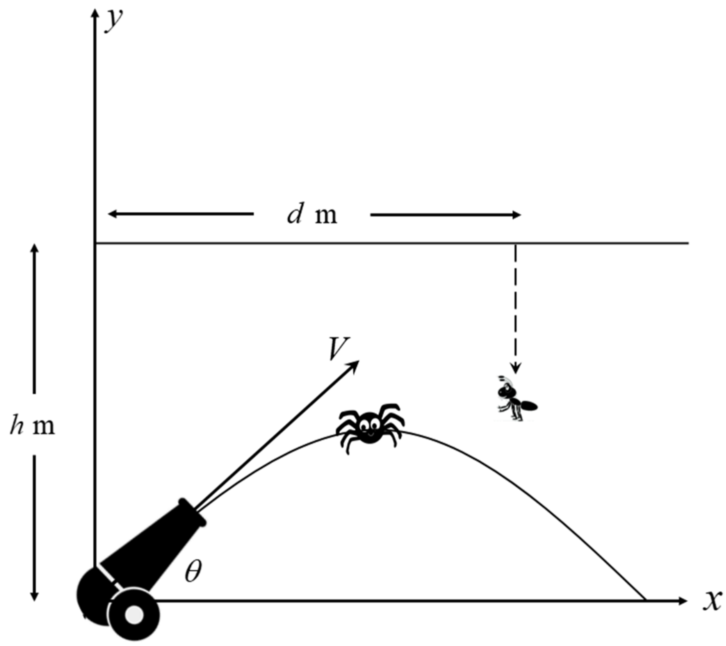
(e) Find the volume of the solid of revolution formed when the region in the first quadrant bounded by the y - axis, $y = \frac{1}{2}x + 3$, and $y = x^2$ is rotated about the y - axis. **3**

End of Question 12

Question 13 (14 marks) Start a new page.

(a) A spider is fired from a cannon and collides mid-air with an ant falling from the ceiling. The cannon is fired at the same moment the ant begins to fall. The ceiling is h metres high, and the horizontal distance between the cannon and the ant is d metres. Let the acceleration due to gravity be g m/s^2.

Let the spider's angle of projection be θ, and let its initial speed be V m/s.



The positions of the spider and the ant respectively are:

$$\underset{\sim}{r}_S = \begin{pmatrix} Vt\cos\theta \\ Vt\sin\theta - \frac{1}{2}gt^2 \end{pmatrix} \qquad \text{Spider}$$

(Do not prove this)

$$\underset{\sim}{r}_A = \begin{pmatrix} d \\ h - \frac{1}{2}gt^2 \end{pmatrix}. \qquad \text{Ant}$$

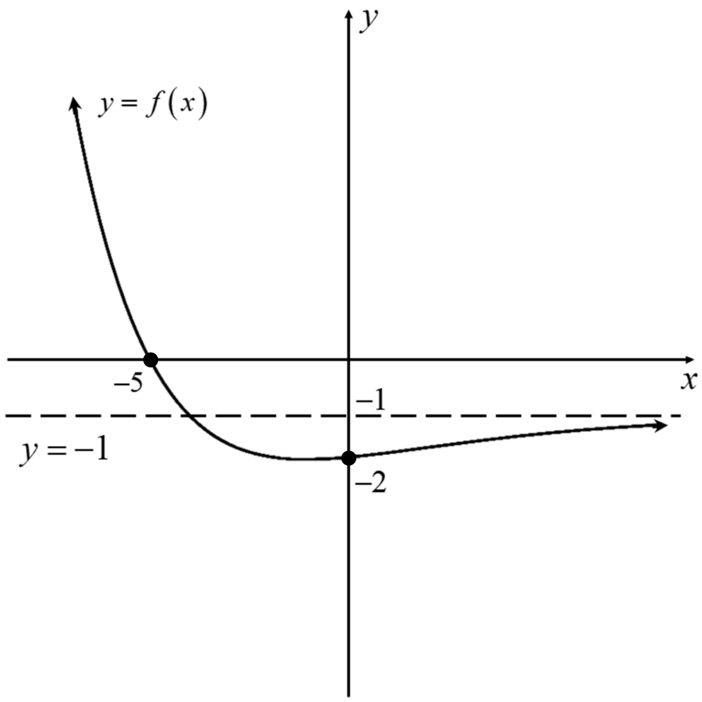
(i) Show that $\tan\theta = \dfrac{h}{d}$. **2**

(ii) Show that $V > \sqrt{\dfrac{g\left(d^2+h^2\right)}{2h}}$. **3**

Question 13 continues on the following page

Question 13 (continued)

(b) Consider the sketch of $y = f(x)$ given below. **2**



Sketch the curve $y = xf(x)$.

(c) (i) Expand $(2y-1)^3$. **1**

(ii) Hence, find the inverse function of $f(x) = 4x^3 - 6x^2 + 3x$. **2**

(d) (i) If $t = \tan\frac{x}{2}$, show that $\sqrt{\frac{1+\sin x}{1-\cos x}} = \frac{1}{\sqrt{2}}\left|\frac{t+1}{t}\right|$. **2**

(ii) Hence, solve $\sqrt{\frac{1+\sin x}{1-\cos x}} = \frac{1}{\sqrt{2}}$ for $-\pi \le x \le \pi$. **2**

End of Question 13

Question 14 (16 marks) Start a new page.

(a) A gaggle of geese grows according to the differential equation **3**

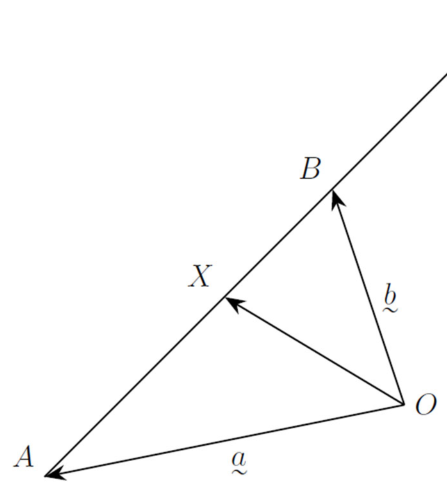
$$\frac{dG}{dt} = \frac{(G+2)(G+3)}{G+4},$$

where G is the number of geese after t days.

Initially, there were 8 geese.

Given that $\dfrac{G+4}{(G+2)(G+3)} = \dfrac{2}{G+2} - \dfrac{1}{G+3}$, find how many days it will take for there to be 1000 geese in the gaggle. Give your answer correct to 3 decimal places.

(b) The point X is chosen within the interval AB such that $AX : XB = m : n$, where m and n are positive and $m \neq n$. Let $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$, as shown.



(i) Show that $\overrightarrow{OX} = \dfrac{n\underset{\sim}{a} + m\underset{\sim}{b}}{m+n}$. **2**

(ii) The point Y is on the line AB such that **2**

$$\overrightarrow{OY} = \frac{n\underset{\sim}{a} - m\underset{\sim}{b}}{m-n} \quad \text{[Do not prove this]}$$

If OX is perpendicular to OY, show that $n|\underset{\sim}{a}| = m|\underset{\sim}{b}|$.

Question 14 continues on the following page

Question 14 (continued)

(c) Use mathematical induction to prove, for all positive integers $n \geq 1$, that **3**

$$\frac{1}{1\times 3}-\frac{1}{2\times 4}+\frac{1}{3\times 5}-\cdots+\frac{(-1)^{n+1}}{n\times(n+2)}=\frac{1}{4}+\frac{(-1)^{n+1}}{2(n+1)(n+2)}.$$

(d) The cubic equation $ax^3+bx^2+cx+d=0$ has one root which is equal to the average of the other two roots. Show that $27a^2d+2b^3=9abc$. **2**

(e) Find the particular solution to the differential equation **2**

$$\frac{dy}{dx}=\frac{\cos x}{\cos y}\,e^{2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)}$$

which passes through the origin.

(Note: In the expression above, e is raised to the power of $2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$.)

(f) Let $n \geq 2$ be a positive integer. Let $P(x)$ be a polynomial of degree at most n with non-negative integer coefficients. Suppose that $P(1)=2n^2$. **2**

Explain, using the pigeonhole principle, why there must be at least one coefficient of $P(x)$ which is at least $2n-1$.

End of task

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2023 X1 Trial Solutions

Q1 B

Q2 C

Q3 B

Q4 D

Q5 A

Q6 C

Q7 B

Q8 A

Q9 B

Q10 C

1 **Question 1**

$$P(-2) = 2(-2)^3 - 3(-2) + 2$$
$$= -16 + 6 + 2$$
$$= -8$$

$\therefore$ (B) ✓

1 **Question 2**

$$\begin{pmatrix}-3\\4\end{pmatrix} \cdot \begin{pmatrix}-1\\-7\end{pmatrix} = 3 - 28$$
$$= -25$$

$$\therefore \quad \cos\theta = \frac{-25}{5 \times 5\sqrt{2}}$$
$$= -\frac{1}{\sqrt{2}}$$
$$\theta = 135°$$

$\therefore$ (C) ✓

1 **Question 3**

$$\int \sin^2 2x \, dx = \frac{1}{2}\int (1 - \cos 4x) \, dx$$
$$= \frac{1}{2}\left(x - \frac{1}{4}\sin 4x\right) + C$$

$\therefore$ (B) ✓

1 **Question 4**

Domain: $-1 \leq 2x \leq 1$

$-\frac{1}{2} \leq x \leq \frac{1}{2}$

Range: $-\frac{3\pi}{2} \leq y \leq \frac{3\pi}{2}$

$\therefore$ (D) ✓

1 **Question 5**

$x = \tan\theta$ When $x = 0$, $\theta = 0$

$dx = \sec^2\theta \, d\theta$ When $x = \sqrt{3}$, $\theta = \frac{\pi}{3}$

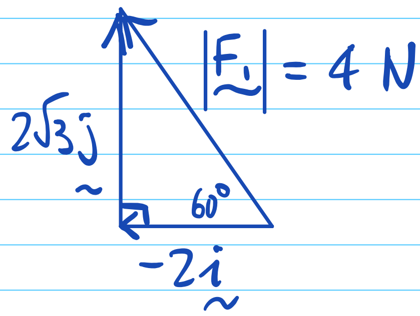
$$\therefore \int_0^{\sqrt{3}} \frac{1}{(x^2+1)^2} \, dx = \int_0^{\frac{\pi}{3}} \frac{\sec^2\theta}{(\tan^2\theta + 1)^2} \, d\theta$$

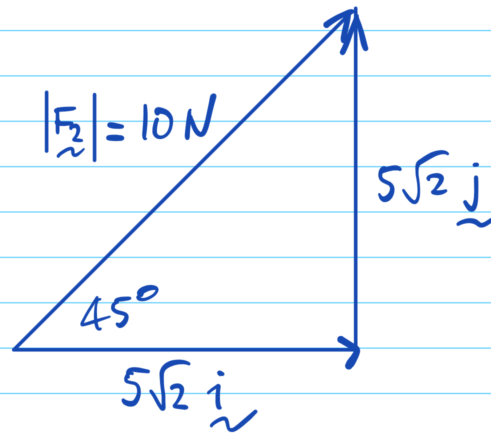
$$= \int_0^{\frac{\pi}{3}} \frac{\cancel{\sec^2\theta}}{(\sec^2\theta)^{\cancel{2}}} \, d\theta$$

$$= \int_0^{\frac{\pi}{3}} \cos^2\theta \, d\theta$$

$\therefore$ (A) ✓

1 **Question 6**





$$\therefore \underset{\sim}{F_1} + \underset{\sim}{F_2} = -2\underset{\sim}{i} + 2\sqrt{3}\,\underset{\sim}{j} + 5\sqrt{2}\,\underset{\sim}{i} + 5\sqrt{2}\,\underset{\sim}{j}$$

$$= (-2+5\sqrt{2})\,\underset{\sim}{i} + (2\sqrt{3}+5\sqrt{2})\,\underset{\sim}{j}$$

$\therefore$ (C) ✓

1 **Question 7**

$x = 2 - e^{-4t}$

$v = 4e^{-4t}$

$a = -16e^{-4t}$

$= -16(2-x)$

$= 16(x-2)$

$\therefore$ (B) ✓

1

Question 8

$\frac{dy}{dx} = 0$ when

$$\frac{x+y}{y-2} = 0$$

$$x+y = 0$$

$$y = -x$$

$\therefore$ Solution curves have gradient zero along the line $y = -x$.

Also, $\frac{dy}{dx}$ is undefined when $y - 2 = 0$

$$y = 2$$

$\therefore$ tangents are vertical along $y = 2$.

$\therefore$ (A) ✓

1

Question 9

Place the four wombats first in $(4-1)! = 3! = 6$ ways (fix one wombat to occupy the top position):

W

_ _

W W

_ _

W

There are 4 spaces between the wombats and no two possums can occupy the same space.

Choose which 3 spaces the possums will occupy in ${}^4C_3$ ways.
Then fill those spaces in $3!$ ways.

$\therefore \quad 3! \times {}^4C_3 \times 3! = 144$ ways

$\therefore$ (B) ✓

1 Question 10

The projection of $\underset{\sim}{u}$ onto $\underset{\sim}{v}$ is

$$\text{proj}_{\underset{\sim}{v}}\, \underset{\sim}{u} = \frac{\underset{\sim}{u} \cdot \underset{\sim}{v}}{|\underset{\sim}{v}|^2}\, \underset{\sim}{v}$$

$$= \frac{\underset{\sim}{u} \cdot \underset{\sim}{v}}{|\underset{\sim}{u}|\,|\underset{\sim}{v}|}\, \underset{\sim}{v} \quad \text{Since } |\underset{\sim}{u}| = |\underset{\sim}{v}|$$

$$= \left(\frac{\underset{\sim}{u}}{|\underset{\sim}{u}|}\right) \cdot \left(\frac{\underset{\sim}{v}}{|\underset{\sim}{v}|}\right) \underset{\sim}{v}$$

$$= \left(\hat{\underset{\sim}{u}} \cdot \hat{\underset{\sim}{v}}\right) \underset{\sim}{v}$$

$\therefore$ (C) ✓

2

Question 11

(a) $$\int_0^{\frac{3}{\sqrt{2}}} \frac{dx}{\sqrt{3^2 - x^2}} = \left[\sin^{-1}\frac{x}{3}\right]_0^{\frac{3}{\sqrt{2}}}$$

$$= \sin^{-1}\frac{1}{\sqrt{2}} - \sin^{-1} 0$$

$$= \frac{\pi}{4}$$

3

(b) $$3\cos x - \sqrt{3}\sin x = R\cos(x + \alpha)$$
$$= R\cos\alpha\cos x - R\sin\alpha \cdot \sin x$$

Equating coefficients:

$$3 = R\cos\alpha \quad (1)$$
$$\sqrt{3} = R\sin\alpha \quad (2)$$

Note:
These equations imply $\sin\alpha$, $\cos\alpha$ are both positive $\therefore$ take α in first quadrant.

$\frac{(2)}{(1)}$: $$\tan\alpha = \frac{1}{\sqrt{3}}$$

$$\alpha = \frac{\pi}{6}$$

$(1)^2 + (2)^2$: $$3^2 + (\sqrt{3})^2 = R^2(\cos^2\alpha + \sin^2\alpha)$$
$$9 + 3 = R^2 \times 1$$
$$R = \sqrt{12} \qquad R > 0$$
$$= 2\sqrt{3}$$

Note: $R \neq -2\sqrt{3}$

$$\therefore 3\cos x - \sqrt{3}\sin x = 2\sqrt{3}\cos\left(x + \frac{\pi}{6}\right)$$

2 (c) General term of $\left(x - \frac{2}{x^2}\right)^6$ is

$$T_{r+1} = \binom{6}{r} x^{6-r} \left(\frac{-2}{x^2}\right)^r$$

$$= \binom{6}{r} x^{6-r} (-2)^r x^{-2r}$$

$$= \binom{6}{r} (-2)^r x^{6-3r}$$

Want coefficient of x^0

$\therefore 0 = 6 - 3r$

$3r = 6$

$r = 2$

$\therefore$ constant term is $\binom{6}{2}(-2)^2 = 60$

2 (d) $A = \pi r^2$

$$\frac{dA}{dr} = 2\pi r$$

$$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$$

Let $r = 8$

$$= 2\pi(8) \times 2$$

$$= 32\pi \text{ cm}^2/\text{s}$$

(e) 1 (i) choose 12 players from a total of 24 players $(8+16=24)$

$$= {}^{24}C_{12}$$

$$= 2704156$$ ✓

2 (ii) $\#(7L, 5R) + \#(8L, 4R)$

$$= \underbrace{{}^{8}C_{7}}_{\text{choose 7 lefties}} \times \underbrace{{}^{16}C_{5}}_{\text{choose 5 righties}} + \underbrace{{}^{8}C_{8}}_{\text{choose 8 Lefties}} \times \underbrace{{}^{16}C_{4}}_{\text{choose 4 righties}}$$ ✓

$$= 36764$$ ✓

3 (f) $\dfrac{3}{3-2x} \geq x+1$

Note: $3-2x \neq 0$

$x \neq \dfrac{3}{2}$

(This was required for full marks)

$$\frac{3}{3-2x} - (x+1) \geq 0$$

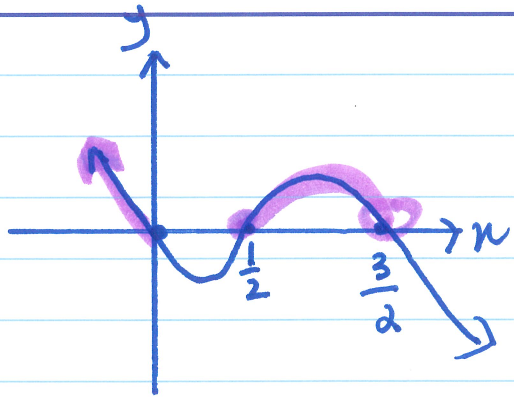
$$\frac{3-(3-2x)(x+1)}{3-2x} \geq 0$$

$$\frac{3-(3x+3-2x^2-2x)}{3-2x} \geq 0$$

$$\frac{2x^2-x}{\cancel{3-2x}} \times (3-2x)^{\cancel{2}} \geq 0 \times (3-2x)^2$$

$$x(2x-1)(3-2x) \geq 0$$

Note: this is a negative cubic



Soln is $\{x : x \leq 0 \ \cup \ \frac{1}{2} \leq x < \frac{3}{2}\}$

Note there is no equal sign here for full marks as $x \neq \frac{3}{2}$.

Some students had trouble with this question. Please note the process in the solution. Always start these questions by moving everything to the L.H.S.

$\boxed{\text{Q12}}$ a/i) $\dfrac{dB}{dt} = -kB$

$$B = B_0 e^{-kt}$$

$$\frac{dB}{dt} = -kB_0 e^{-kt}$$

$$= -kB$$

Thus $B = B_0 e^{-kt}$ is a solution of the differential equation $\frac{dB}{dt} = -kB$

a/ii) At $t = 7$, $B = \frac{1}{100} B_0$

$$\frac{1}{100} B_0 = B_0 e^{-k \times 7}$$

AW 1 Mark for Subs $B = \frac{1}{100} B_0$

$$\frac{1}{100} = e^{-7k}$$

$$-7k = \ln\left(\frac{1}{100}\right)$$

$$k = \frac{\ln(1/100)}{-7} = -\frac{\ln(0.01)}{7} \text{ OR } = \frac{1}{7}\ln(100)$$

b)

$$x = 3\cos\theta \quad ①$$

$$y = 3\sin\theta - 5 \quad ②$$

$$y + 5 = 3\sin\theta \quad ②$$

$$①^2 + ②^2: \quad x^2 + (y+5)^2 = 9(\sin^2\theta + \cos^2\theta)$$

$$x^2 + (y+5)^2 = 9$$

$\boxed{1M}$: Attempting the substitution to eliminate θ.

$\boxed{Q12}$

c/i) $\frac{1}{2}\left[\sin(2\theta+\theta) + \sin(2\theta-\theta)\right] = \sin 2\theta . \cos\theta$

$\sin 3\theta + \sin\theta = 2\sin 2\theta . \cos\theta$ $\boxed{1M}$

OR Equivalent product.

c/ii) $\dfrac{\sin\theta + \sin 3\theta}{\cos 2\theta + 1} = 2\sin\theta$

$LHS = \dfrac{2\sin 2\theta . \cos\theta}{\cos^2\theta - \sin^2\theta + \sin^2\theta + \cos^2\theta}$ ✓

$= \dfrac{4\sin\theta\cos^2\theta}{2\cos^2\theta}$

$= 2\sin\theta = RHS$ ✓

d) $\displaystyle\int \frac{x-2}{\sqrt{x+2}}$ let $u = x+2 \longrightarrow x = u-2$

$\dfrac{du}{dx} = 1 \quad \therefore du = dx$ ✓

$= \displaystyle\int \frac{u-2-2}{\sqrt{u}}\, du$

$= \displaystyle\int (u^{\frac{1}{2}} - 4u^{-\frac{1}{2}})\, du$ ✓

$= \dfrac{2u^{3/2}}{3} - 4\sqrt{u} + c$

$= \dfrac{2(x+2)^{3/2}}{3} - 4\sqrt{x+2} + c$ ✓

Final Ans must be in terms of x not u.

Q12

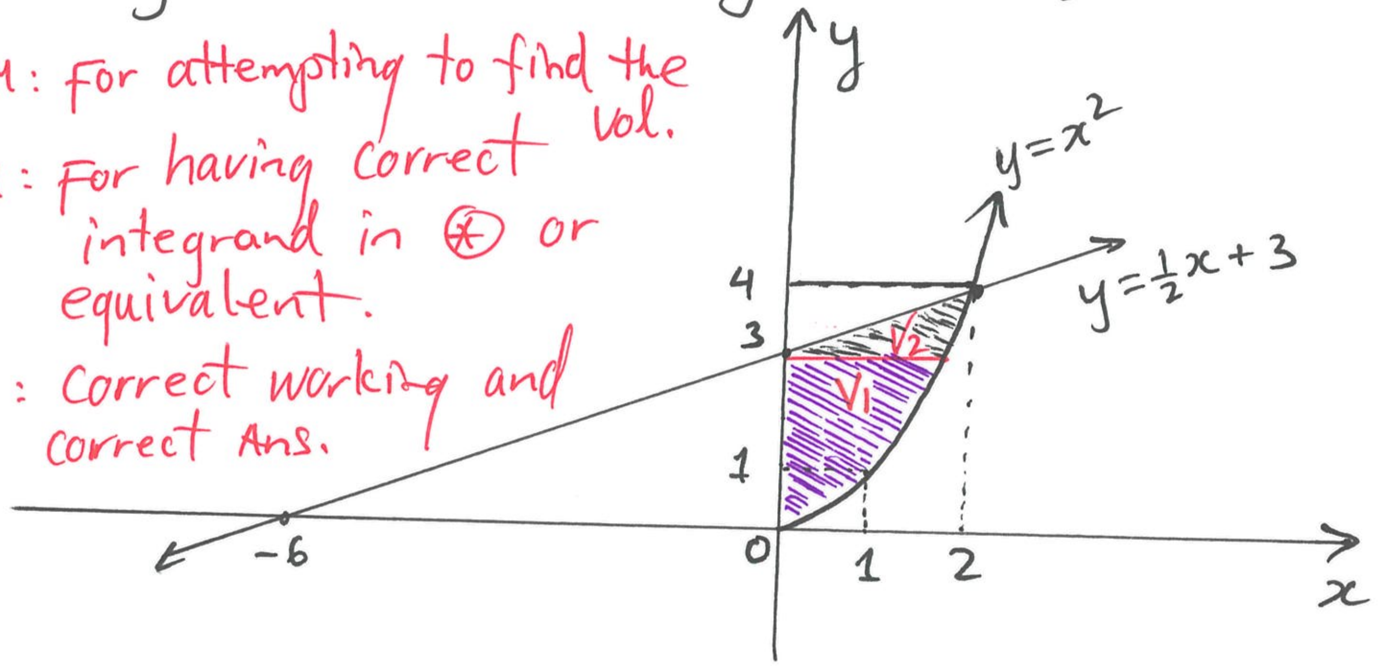
e) $y = \frac{1}{2}x + 3 \longrightarrow x = 2y - 6$

$y = x^2 \longrightarrow x = \sqrt{y} \quad (x > 0)$

1M: For attempting to find the vol.

2M: For having correct integrand in ⊛ or equivalent.

3M: Correct working and correct Ans.



$V = V_1 + V_2$

$= \pi \int_0^3 y\,dy + \pi \int_3^4 \left(y - (2y-6)^2\right) dy$ ⊛

$= \pi \left[\frac{y^2}{2}\right]_0^3 + \pi \int_3^4 (-4y^2 + 25y - 36)\,dy$

$= \frac{9\pi}{2} + \pi \left[\frac{-4y^3}{3} + \frac{25y^2}{2} - 36y\right]_3^4$

$= \frac{9\pi}{2} + \pi \left[\frac{-4(4)^3}{3} + \frac{25(4)^2}{2} - 36(4) - \left(\frac{-4\times3^3}{3} + \frac{25\times3^2}{2} - 36\times3\right)\right]$

$= \frac{9\pi}{2} + \pi \left[\frac{13}{6}\right] = \frac{20\pi}{3}$ cubic units.

Q13)

a) i) Show $\tan\theta = \frac{h}{d}$

Spider & Ant collide when position is <u>equivalent</u>.

$x: \quad Vt\cos\theta = d$ 1

$$t = \frac{d}{V\cos\theta} \quad (\star)$$

$y: \quad Vt\sin\theta - \cancel{\tfrac{1}{2}gt^2} = h - \cancel{\tfrac{1}{2}gt^2}.$

$$h = Vt\sin\theta$$

sub $(\star) \Rightarrow h = \cancel{V}\left(\frac{d}{\cancel{V}\cos\theta}\right)\sin\theta$

$\therefore \tan\theta = \frac{h}{d}$ as required. 1

Comments

Many students assumed the identity was true by making the assumption that the cannon was aiming at the ant. Instead, students had to <u>substitute</u> $x = d$ & $y = h$ to prove it.

ii) Show $V > \sqrt{\frac{g(d^2+h^2)}{2h}}$

$\boxed{\text{Solution 1: Using } x_S > x_A}$

i.e spider will land further away than ant.

$\Rightarrow \boxed{x_S > x_A}$ where x is the horizontal distance

$\boxed{x_A = d} \to$ Fixed constant.

For x_S,

$$\underset{\sim}{V}_S = \begin{pmatrix} V\cos\theta \\ V\sin\theta - gt \end{pmatrix} \qquad \underset{\sim}{V}_{S,y} = 0 \Rightarrow V\sin\theta = gt$$

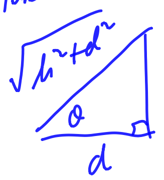
y component

$$t = \frac{V\sin\theta}{g}. \quad 1$$

$t = \frac{2V\sin\theta}{g} \Rightarrow x_S = 2V\left(\frac{V\sin\theta}{g}\right)\cos\theta.$

$x_S > x_A \Rightarrow \frac{2V^2\sin\theta\cos\theta}{g} > d$

$\tan\theta = \frac{h}{d}$ 1



$$V^2 > \frac{gd}{2\sin\theta\cos\theta}$$

$$\Rightarrow V^2 > \frac{g\cancel{d}}{2\frac{h}{\sqrt{h^2+d^2}}\frac{\cancel{d}}{\sqrt{h^2+d^2}}} \Rightarrow V > \sqrt{\frac{g(h^2+d^2)}{2h}} \quad 1$$

as required.

Comments

Poorly done. Many students derived the cartesian form of the equation but failed to substitute points or conditions into the equation of motion to form the identity. Students also lost marks for equations/inequalities that were not valid.

Solution 2 : Using conditions at p.t of impact.

At p.t of impact,

$\overset{\rightharpoonup}{\sim} S, x = d \Rightarrow V t \cos\theta = d$ ★

$\overset{\rightharpoonup}{\sim} A, y > 0$ (in air)

} condition holds simultaneously.

$\Rightarrow h - \frac{1}{2} g t^2 > 0$

$-\frac{1}{2} g t^2 > -h$

$t^2 < \frac{2h}{g}$

$t < \sqrt{\frac{2h}{g}}$ (1)

as $\boxed{t > 0}$

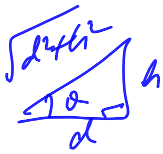
From ★

$V t \cos\theta = d$

$V = \frac{d}{t \cos\theta}$

$V = \frac{d}{\cos\theta} \cdot \frac{1}{t}$

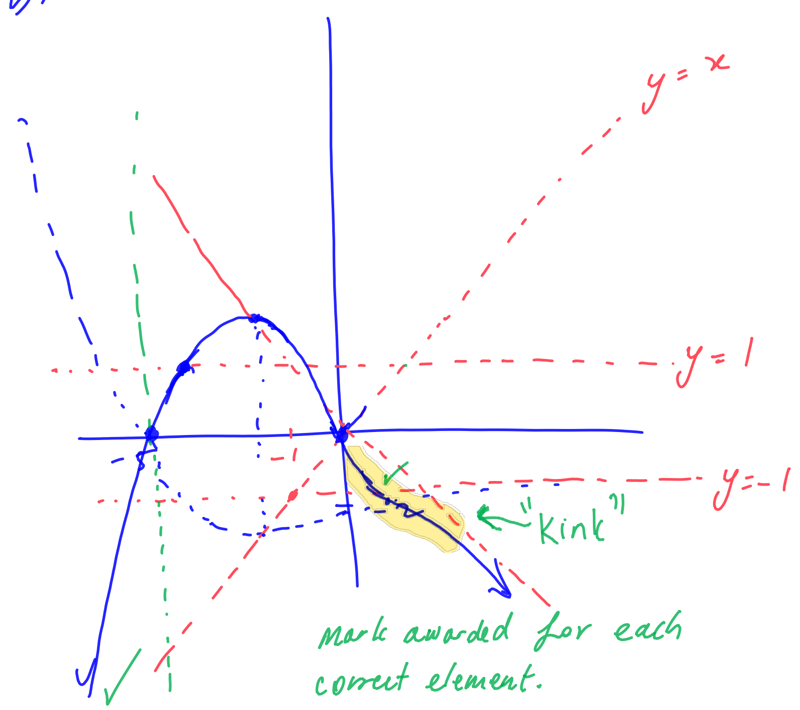
$V > \frac{d}{\cos\theta} \sqrt{\frac{2h}{g}}$ (1) as $\frac{1}{t} > \sqrt{\frac{g}{2h}}$



$V > \frac{d}{\frac{d}{\sqrt{d^2+h^2}} \sqrt{\frac{2h}{g}}}$

$\therefore V > \sqrt{\frac{g(d^2+h^2)}{2h}}$ (1)

b)



Mark awarded for each correct element.

Note:

$- \times - = +$

$- \times + = -$

$x \times 1^+ = x^+$ (above $y=x$)

similarly,

$x \times 1^- = x^-$ (below $y=x$)

$0 \times$ anything $= 0 \rightarrow$ Becomes x-int.

number $\times \pm = \pm$ number.

Comments

Only a few students obtained full marks. Students had to have the "kink" on the right. Failing to have this resulted in a mark deducted.

c) i) $(2y-1)^3 = (2y)^3 - 3(2y)^2 + 3(2y) - 1$ (1)

$= 8y^3 - 12y^2 + 6y - 1$

ii) $f(x) = 4x^3 - 6x^2 + 3x$

interchanging x & y:

$4y^3 - 6y^2 + 3y = x$

$8y^3 - 12y^2 + 6y = 2x$ (1) Working towards

$8y^3 - 6y^2 + 6y - 1 = 2x - 1$ (2) Correct Sol^n.

$\Rightarrow (2y-1)^3 = 2x - 1$ from i)

$2y - 1 = \sqrt[3]{2x-1}$

$\therefore y = \dfrac{1 + \sqrt[3]{2x-1}}{2}$ i.e $f^{-1}(x) = \dfrac{1+\sqrt[3]{2x-1}}{2}$

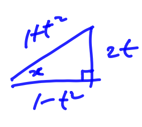
Comments

Most students obtained at least 1 mark. There were a number of students that did not interchange x/y for the inverse function at all which resulted in zero marks being awarded.

d) i) $t = \tan\frac{x}{2}$ show $\sqrt{\dfrac{1+\sin x}{1-\cos x}} = \dfrac{1}{\sqrt{2}}\left|\dfrac{t+1}{t}\right|$

Done well.

$LHS = \sqrt{\dfrac{1 + \frac{2t}{1+t^2}}{1 - \frac{1-t^2}{1+t^2}}}$ (1) Working towards



(2) Correct Sol^n.

$= \sqrt{\dfrac{\frac{1+t^2+2t}{\cancel{1+t^2}}}{\frac{\cancel{1}+t^2-(\cancel{1}-t^2)}{\cancel{1+t^2}}}} = \sqrt{\dfrac{(t+1)^2}{2t^2}}$

$= \left|\dfrac{t+1}{t}\right|\dfrac{1}{\sqrt{2}}$ as required.

ii) $\sqrt{\dfrac{1+\sin x}{1-\cos x}} = \dfrac{1}{\sqrt{2}}$

from i) $\Rightarrow \left|\dfrac{t+1}{t}\right|\dfrac{1}{\sqrt{2}} = \dfrac{1}{\sqrt{2}}$ $\quad -\pi \le x \le \pi$

$\left|\dfrac{t+1}{t}\right| = 1$ $\quad$ ① Working forwards

$\left|1+\dfrac{1}{t}\right| = 1$ $\quad$ ② Correct sol^n.

$1+\dfrac{1}{t} = \pm 1$

$\dfrac{1}{t} = -1 \pm 1$

$\dfrac{1}{t} = -2, 0$

$\Rightarrow \cot\left(\dfrac{x}{2}\right) = -2, 0 \qquad -\dfrac{\pi}{2} \le \dfrac{x}{2} \le \dfrac{\pi}{2}.$

$\therefore \tan\dfrac{x}{2} = -\dfrac{1}{2}$, $\tan\dfrac{x}{2} = $ No sol^n. $\Rightarrow \dfrac{x}{2} = -\dfrac{\pi}{2}, \dfrac{\pi}{2}$

$x = -\pi, \pi$

$\Rightarrow \dfrac{x}{2} = -\tan^{-1}\left(\dfrac{1}{2}\right)$

✓S	A
T	C✓

$\therefore x = -2\tan^{-1}\left(\dfrac{1}{2}\right), \pi, -\pi$

$\doteqdot -0.93$

Comments

Only a handful managed to get 2 marks. Students had to obtain the correct equations and test $x = \pm\pi$ to obtain full marks. Many students unnecessarily squared their absolute equations which overcomplicated the algebra.

<u>Question 14</u>

3

(a) $\dfrac{dG}{dt} = \dfrac{(G+2)(G+3)}{G+4}$

$$\therefore \int_0^t dt = \int_8^G \frac{G+4}{(G+2)(G+3)}\, dG \quad ✓$$

$$[t]_0^t = \int_8^G \left(\frac{2}{G+2} - \frac{1}{G+3}\right) dG$$

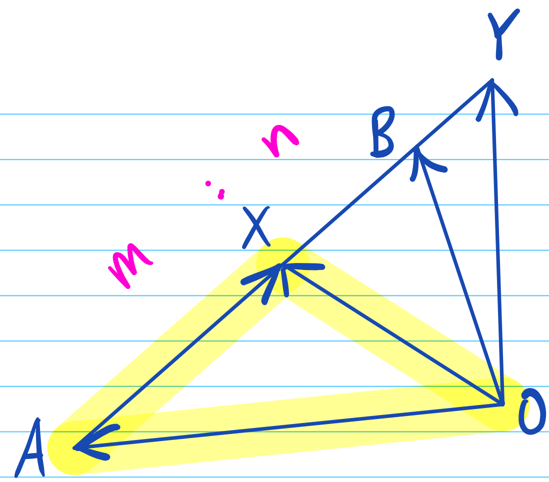
$$t = \Big[2\ln|G+2| - \ln|G+3|\Big]_8^G \quad ✓$$

$$t = 2\ln(G+2) - \ln(G+3) - (2\ln 10 - \ln 11)$$

Sub in $G = 1000$:

$t \doteqdot 4.701$ days (3 d.p.) ✓

(b)



2 (i) Consider ΔOAX:

$$\overrightarrow{OX} = \overrightarrow{OA} + \overrightarrow{AX}$$

$$= \underset{\sim}{a} + \frac{m}{m+n}\overrightarrow{AB}$$

$$= \underset{\sim}{a} + \frac{m}{m+n}(\underset{\sim}{b} - \underset{\sim}{a})$$

$$= \frac{(\cancel{m}+n)\underset{\sim}{a} + m\underset{\sim}{b} - \cancel{m\underset{\sim}{a}}}{m+n}$$

$$= \frac{n\underset{\sim}{a} + m\underset{\sim}{b}}{m+n}$$

2 (ii) Since OX is perpendicular to OY,

$$\overrightarrow{OX} \cdot \overrightarrow{OY} = 0$$

$$\therefore \left(\frac{n\underset{\sim}{a} + m\underset{\sim}{b}}{m+n}\right) \cdot \left(\frac{n\underset{\sim}{a} - m\underset{\sim}{b}}{m-n}\right) = 0$$

$$\frac{(n\underset{\sim}{a}+m\underset{\sim}{b})\cdot(n\underset{\sim}{a}-m\underset{\sim}{b})}{(m+n)(m-n)}=0$$

$$n^2\underset{\sim}{a}\cdot\underset{\sim}{a}-\cancel{mn\underset{\sim}{a}\cdot\underset{\sim}{b}}+\cancel{mn\underset{\sim}{a}\cdot\underset{\sim}{b}}-m^2\underset{\sim}{b}\cdot\underset{\sim}{b}=0$$

$$n^2\underset{\sim}{a}\cdot\underset{\sim}{a}-m^2\underset{\sim}{b}\cdot\underset{\sim}{b}=0$$

$$n^2|\underset{\sim}{a}|^2=m^2|\underset{\sim}{b}|^2$$

$$n|\underset{\sim}{a}|=m|\underset{\sim}{b}|$$

3 (c) <u>Base case</u>:

For $n=1$, $LHS=\frac{1}{1\times3}$

$$=\frac{1}{3}$$

$$RHS=\frac{1}{4}+\frac{(-1)^2}{2(1+1)(1+2)}$$

$$=\frac{1}{4}+\frac{1}{12}$$

$$=\frac{1}{3}$$

$$=LHS$$

$\therefore$ True for $n=1$.

Assume true for $n=k$:

$$\frac{1}{1\times3}-\frac{1}{2\times4}+\frac{1}{3\times6}-\cdots+\frac{(-1)^{k+1}}{k(k+2)}=\frac{1}{4}+\frac{(-1)^{k+1}}{2(k+1)(k+2)} \quad (*)$$

Prove true for $n=k+1$:

R.T.P.

$$\underbrace{\frac{1}{1\times3}-\frac{1}{2\times4}+\cdots+\frac{(-1)^{k+1}}{k(k+2)}}_{\text{use }(*)}+\frac{(-1)^{k+2}}{(k+1)(k+3)}=\frac{1}{4}+\frac{(-1)^{k+2}}{2(k+2)(k+3)}$$

$$LHS=\frac{1}{4}+\frac{(-1)^{k+1}}{2(k+1)(k+2)}+\frac{(-1)^{k+2}}{(k+1)(k+3)}$$

$$=\frac{1}{4}+\frac{(-1)^{k+2}}{2(k+1)}\left[-\frac{1}{k+2}+\frac{2}{k+3}\right]$$

$$=\frac{1}{4}+\frac{(-1)^{k+2}}{2(k+1)}\left[\frac{-k-3+2k+4}{(k+2)(k+3)}\right]$$

$$=\frac{1}{4}+\frac{(-1)^{k+2}}{2\cancel{(k+1)}}\times\frac{\cancel{k+1}}{(k+2)(k+3)}$$

$$=\frac{1}{4}+\frac{(-1)^{k+2}}{2(k+2)(k+3)}$$

$=RHS$

$\therefore$ True by induction.

Note: $\frac{2\alpha + 2\beta}{2} = \alpha + \beta$, so $\alpha + \beta$ is the average of 2α, 2β.

2 (d) $ax^3 + bx^2 + cx + d = 0$

Let the roots be 2α, 2β, $\alpha + \beta$.

$$2\alpha + 2\beta + \alpha + \beta = -\frac{b}{a}$$ sum of roots

$$3(\alpha + \beta) = -\frac{b}{a}$$

$$\alpha + \beta = \frac{-b}{3a}$$ ✓

But $\alpha + \beta$ is a root, so we can sub $x = \frac{-b}{3a}$ into the cubic equation:

$$a\left(\frac{-b}{3a}\right)^3 + b\left(\frac{-b}{3a}\right)^2 + c\left(\frac{-b}{3a}\right) + d = 0$$

$$\frac{-\cancel{a}b^3}{27a^{\cancel{3}2}} + \frac{b^3}{9a^2} - \frac{bc}{3a} + d = 0$$

$$-b^3 + 3b^3 - 9abc + 9a^2d = 0$$

$$27a^2d + 2b^3 = 9abc$$ ✓

2 (e) Use product-to-sum identity:

$$\frac{dy}{dx} = \frac{\cos x}{\cos y} e^{2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)}$$

$$\frac{dy}{dx} = \frac{\cos x}{\cos y} e^{\sin x + \sin y} \checkmark$$

$$\int_0^y e^{-\sin y} \cos y \, dy = \int_0^x \cos x \, e^{\sin x} dx$$

$$\left[-e^{-\sin y}\right]_0^y = \left[e^{\sin x}\right]_0^x$$

$$-e^{-\sin y} + 1 = e^{\sin x} - 1$$

$$-e^{-\sin y} = e^{\sin x} - 2 \checkmark$$

$$-\sin y = \ln\left(2 - e^{\sin x}\right)$$

$$y = -\arcsin\left(\ln\left(2 - e^{\sin x}\right)\right)$$

2

(f) Let $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$.

Then $P(1) = a_n + a_{n-1} + \cdots + a_1 + a_0$

$\therefore a_n + a_{n-1} + \cdots + a_1 + a_0 = 2n^2$.

A total of $2n^2$ needs to be distributed amongst the $n+1$ coefficients.

There are $2n^2$ pigeons going into $n+1$ pigeonholes, namely the coefficients of $P(x)$.

By the pigeonhole principle, there is at least one coefficient which is at least $\dfrac{2n^2}{n+1}$. ✓

However, by long division or by inspection:

$$\frac{2n^2}{n+1} = 2n - 2 + \frac{2}{n+1}, \text{ where } n \geq 2.$$

Since $0 < \dfrac{2}{n+1} < 1$, we may round up to the next integer, namely:

$2n - 2 + 1 = 2n - 1$.

Therefore, there is at least one coefficient which is at least $2n-1$. ✓

Q14 comments

(a) Quite well done. Most students were able to separate and solve the differential equation. There were some calculation errors.

(b)(i) Quite well done. Students were awarded partial marks for recognising a suitable path, e.g. $\overrightarrow{OX} = \overrightarrow{OA} + \overrightarrow{AX}$ OR $\overrightarrow{OX} = \overrightarrow{OB} + \overrightarrow{BX}$ and attempting to use the given ratio.

(b)(ii) Students struggled with this part. The main misconception was that $\underset{\sim}{a} \cdot \underset{\sim}{a} = \underset{\sim}{a}^2$. Note that vectors can not be squared. The correct statement is $\underset{\sim}{a} \cdot \underset{\sim}{a} = |\underset{\sim}{a}|^2$.

(c) Well done. The most common error was writing:

$$\boxed{\frac{1}{4} - \frac{(-1)^{k+1}}{2(k+1)(k+2)} + \frac{(-1)^{k+2}}{(k+1)(k+3)}}$$

where the incorrect sign is circled in red. However, often when this error was made, students were still able to show their understanding of the correct method (factorising) and get 2 out of 3 marks.

(d) Students found this part challenging. The correct approach was mostly attempted, but few students were able to obtain the required identity.

(e) Quite well done. Some students used the product-to-sum identity and attempted to integrate, obtaining 1 mark out of 2. Common errors included solving $c = 0$ instead of $c = -2$, and forgetting the negative signs, circled below:

$$\boxed{-e^{-\sin y} = e^{\sin x} - 2}$$

(f) This was the most difficult part of the paper. Very few students were awarded one mark out of two. Some students wrote valid proofs by contradiction which could not be awarded any marks, since the question required the pigeonhole principle to be used.